Noether symmetry approach to scalar-field-dominated cosmology with dynamically evolving G and Λ

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Abstract

This paper studies the cosmological equations for a scalar field φ in the framework of a quantum gravity modified Einstein–Hilbert Lagrangian where G and Λ are dynamical variables. It is possible to show that there exists a Noether symmetry for the point Lagrangian describing this scheme in a FRW universe. Our main result is that the Noether Symmetry Approach fixes both $\Lambda = \Lambda(G)$ and the potential $V = V(\varphi)$ of the scalar field. The method does not lead, however, to easily solvable equations, by virtue of the higher dimensionality of the *reduced configuration space* involved, the additional variable being the running Newton coupling.

I. INTRODUCTION

In recent times, substantial evidence was found for the nonperturbative renormalizability of Quantum Einstein Gravity (see for a review [1]) according with the asymptotic safety scenario [2]. The theory emerging from this construction is not a quantization of classical general relativity. Instead, its bare action corresponds to a nontrivial fixed point of the Renormalization Group (hereafter, RG) flow and is a *prediction* therefore, and not, as usually in quantum field theory, an ad hoc assumption defining some "model". On the other hand, it turns out that G and Λ are then dependent on the characteristic energy scale k at which the physics is probed. The relevant question is how to couple to renormalization group evolution based on the running of k with the spacetime dynamics.

In particular in Ref. [3], an Arnowitt–Deser–Misner (ADM) formulation of "renormalization group (RG) induced" *quantum Einstein gravity* has been presented, building a modified action functional which reduces to the Einstein–Hilbert action when G is constant. Actually, the RG-improved framework characterizes the (dimensionless) running cosmological term $\lambda(k)$ and running Newton parameter $g(k)$, starting from an ultraviolet attractive fixed point [4, 5, 6, 7, 8, 9, 10, 11, 12]. It is there possible to find an explicit k-dependence of the running Newton term $G(k)$ and the running cosmological term $\Lambda(k)$, which is interesting in the attempt of understanding the cosmic era immediately after the big bang as well as the structure of black hole singularity [13, 14, 15]. In order to obtain the RG-improved Einstein equations for a homogeneous and isotropic universe, it is possible to identify k with the inverse of cosmological time, $k \propto 1/t$ [13, 16]. Thus, a dynamical evolution for $G(k)$ and $\Lambda(k)$ induced by their RG running is derived.

In both a pure gravity regime and a massless φ^4 theory in a homogeneous and isotropic space-time, within the framework of the ADM formulation, it is possible to obtain a power-law growth of the scale factor, in full agreement with what is already known on fixed-point cosmology [3]. In Ref. [17] we have also proposed solutions for the pure gravity case derived by means of the so-called *Noether Symmetry Approach*. This method implements a change of variable that usually leads to exact and general solutions of the cosmological equations [18, 19]. The solutions found in Ref. [17] predict that an empty (pure gravity) universe is undergoing an accelerated stage, and hence are well mimicking inflation without need to introduce a scalar field in the cosmic content. The Noether Symmetry Approach has

proved useful also in deriving exact and general solutions for the cosmological equations in a matter-dominated era [20]. They, too, are power-law.

Here, we analyze a scalar-field-dominated cosmology with variable G and Λ within the same framework, still performing the customary procedure prescribed by that approach. Let us immediately point out that the situation we describe is thus seen in a completely different way with respect to that of the massless φ^4 case studied in Ref. [3], by virtue of the peculiar method worked out here. The existence of a Noether symmetry for the Lagrangian (seen as a point Lagrangian on the *reduced configuration space* with coordinates a, G and φ) can in principle be used to obtain a transformed form of the cosmological equations, which often turned out to be solvable in an exact and general way. This method is also interesting by itself, since it leads for consistency to naturally adopting peculiar and original forms (with respect to those usually present in the literature) of the functions $\Lambda = \Lambda(G)$ and $V = V(\varphi)$. This of course encourages future and more refined related work.

In what follows, we first introduce the scalar-field formulation for the RG-improved Einstein equations in section 2, and then apply the Noether Symmetry Approach in section 3. Section 4 studies a fixed-point solution, while section 5 is devoted to some concluding remarks on the very presence of a scalar field in the RG-improved cosmology.

II. LAGRANGIAN WITH VARIABLE G AND Λ

Following Ref. [3], the quantum gravity modified Lagrangian in a Friedmann–Lemaitre–Robertson–Walker universe is taken to be

$$L_{\mathrm{pg}} = \frac{1}{8\pi G}\left(-3a\dot{a}^2 + 3\mathcal{K}a - a^3\Lambda + \frac{1}{2}\mu a^3\frac{\dot{G}^2}{G^2}\right), \qquad (2.1)$$

where G and Λ are functions of time, $\mathcal{K}$ is $-1, 0, 1$ for open, spatially flat and closed universes, respectively, and $\mu \neq 0$ represents a dimensionless interaction parameter without any observational constraint, since it is substantially different from zero only for modifications of general relativity occurring in the very early universe; dots indicate time derivatives. In order to write the *full* Lagrangian L of the theory we add $L_m \equiv L_\varphi \equiv a^3[\dot{\varphi}^2/2 - V(\varphi)]$ to L_{pg}. On the *reduced configuration space* with coordinates (a, G, φ), L thus takes the form

$$L = L(a, G, \varphi) = \frac{1}{8\pi G}\left[-3a\dot{a}^2 + 3\mathcal{K}a - a^3\Lambda + \frac{1}{2}\mu a^3\frac{\dot{G}^2}{G^2} + 4\pi Ga^3\dot{\varphi}^2 - 8\pi Ga^3V(\varphi)\right]. \quad (2.2)$$

3

It is very important to realize that, with respect to the majority of the work previously done in the Noether Symmetry Approach, we are here in the presence of a more involved picture, mainly by virtue of the three-dimensionality of the *reduced configuration space*. A similar peculiar property can, anyway, be found already when dealing with Bianchi universes [21], where the appropriate configuration space consists in fact of four variables. Nevertheless, in that case, this feature allows immediate exact integration, although only in simple cases. (In general, however, the number of symmetries is often sufficiently high to permit a good reduction of the configuration space, which indeed seems to be physically interesting, but a complete analysis has not been done.) On the other hand, in Ref. [22], which investigates the behaviour of a homogeneous, anisotropic, and spatially flat universe filled in only with a scalar field φ, three Noether symmetries are found for any $V(\varphi)$, actually independent of the presence of such φ. Exact integration is then possible only when $V(\varphi)$ is a constant. (When such a constant vanishes, a Kasner solution is indeed obtained, while otherwise the expressions of the expansion rates show asymptotic isotropization resulting from the scalar field itself.) A non-trivial positive potential does not lead to exact integration, but it anyhow leads to physically interesting results since it introduces a necessary non-inflationary initial expansion of the universe, still allowing a later inflationary stage.

In the present setting we find that, although the increased number of degrees of freedom is now limited to three, this anyway gives rise to more technical difficulties, which seem extremely hard to overcome at the moment, apparently forbidding a deeper physical discussion. However, our considerations are of course far from being exhaustive, and more work is in order.

III. NOETHER SYMMETRY APPROACH

It is possible to show that there still exists a Noether symmetry for the Lagrangian L of section 2 describing a scalar field coupled to RG-improved Einstein gravity. For this purpose, we consider the vector field

$$X \equiv \alpha(a, G, \varphi)\frac{\partial}{\partial a} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \beta(a, G, \varphi)\frac{\partial}{\partial G} + \dot{\beta}\frac{\partial}{\partial \dot{G}} + \gamma(a, G, \varphi)\frac{\partial}{\partial \varphi} + \dot{\gamma}\frac{\partial}{\partial \dot{\varphi}}, \qquad (3.1)$$

with α, β and γ generic C^1 functions, and

$$\dot{\alpha} \equiv d\alpha/dt = (\partial\alpha/\partial a)\dot{a} + (\partial\alpha/\partial G)\dot{G} + (\partial\alpha/\partial \varphi)\dot{\varphi},$$

4

$$\dot{\beta} \equiv d\beta/dt = (\partial\beta/\partial a)\dot{a} + (\partial\beta/\partial G)\dot{G} + (\partial\beta/\partial\varphi)\dot{\varphi},$$

$$\dot{\gamma} \equiv d\gamma/dt = (\partial\gamma/\partial a)\dot{a} + (\partial\gamma/\partial G)\dot{G} + (\partial\gamma/\partial\varphi)\dot{\varphi}.$$

As in Refs. [17, 20], the condition

$$\mathcal{L}_X L = 0 \tag{3.2}$$

(where $\mathcal{L}_X L$ denotes the Lie derivative of L along X) now leads to a system of partial differential equations for $\alpha = \alpha(a, G, \varphi)$, $\beta = \beta(a, G, \varphi)$, $\gamma = \gamma(a, G, \varphi)$. In the resulting complicated set of equations, the potential $V = V(\varphi)$ and the cosmological term $\Lambda = \Lambda(G)$ occur together with their first derivatives in such a way that the solution of the system of equations determines completely, by itself, their functional forms.

We indeed find

$$X = M \left[a\frac{\partial}{\partial a} + \dot{a}\frac{\partial}{\partial\dot{a}} + 3G\frac{\partial}{\partial G} + 3\dot{G}\frac{\partial}{\partial\dot{G}} + \left(\gamma_0 - \frac{3}{2}\varphi\right)\frac{\partial}{\partial\varphi} - \frac{3}{2}\dot{\varphi}\frac{\partial}{\partial\dot{\varphi}} \right], \tag{3.3}$$

with M and γ_0 arbitrary constants. We can set M to 1 without loss of generality. The choice $\gamma_0 \neq 0$ gives a translation of φ which is clearly inessential, so that we set $\gamma_0 = 0$.

Consistency also requires that the universe is spatially flat, $\mathcal{K} = 0$, and that the μ parameter assumes a fixed value

$$\mu = \frac{2}{3}, \tag{3.4}$$

while the expressions of Λ and V are determined as

$$\Lambda(G) = \Lambda_0 - \frac{\lambda_0}{3}G, \tag{3.5}$$

$$V(\varphi) = \frac{\lambda_0}{24\pi} + 9\lambda_1\varphi^2, \tag{3.6}$$

Λ_0, λ_0 and λ_1 being other arbitrary constants (whose values cannot be treated as easily as before). The energy function associated with L is now

$$
\begin{aligned}
E_L &\equiv \frac{\partial L}{\partial\dot{a}}\dot{a} + \frac{\partial L}{\partial\dot{G}}\dot{G} + \frac{\partial L}{\partial\dot{\varphi}}\dot{\varphi} - L \\
&= \frac{1}{8\pi G}\left[-3a\dot{a}^2 + a^3\Lambda + \frac{1}{2}\mu a^3\frac{\dot{G}^2}{G^2} + 4\pi Ga^3\dot{\varphi}^2 + 8\pi Ga^3 V(\varphi) \right],
\end{aligned}
\tag{3.7}
$$

and, as usual, we have to set $E_L = 0$ to get the first-order Friedmann equation.

It is easy to see that we can perform a change of variables $(a, G, \varphi) \rightarrow (u, v, w)$ implying, say, $X \rightarrow X' = \partial/\partial u$ and $L \rightarrow L'$, from which $\mathcal{L}_{X'}L' = \partial L'/\partial u = 0$, i.e., such that among

5

the new variables there exists a cyclic coordinate u for the transformed Lagrangian L'. From the contractions $i_X du = 1$, $i_X dv = 0$, and $i_X dw = 0$ [19], we in fact find, as a possible choice,

$$u = u(a, G, \varphi) = \ln(a), \tag{3.8}$$

$$v = v(a, G, \varphi) = a^{k_1} G^{k_2/3} \varphi^{-2k_3/3}, \tag{3.9}$$

$$w = w(a, G, \varphi) = a^{k_1'} G^{k_2'/3} \varphi^{-2k_3'/3}, \tag{3.10}$$

k_i and k_i' ($i = 1, 2, 3$) being arbitrary constants such that

$$k_1 + k_2 + k_3 = k_1' + k_2' + k_3' = 0, \tag{3.11}$$

$$k_2 k_3' - k_2' k_3 \neq 0. \tag{3.12}$$

The last constraint is equivalent to assume invertibility of transformations in Eqs. (3.8), (3.9) and (3.10), since it ensures that the Jacobian of the transformation does not vanish.

In order to derive the transformed Lagrangian L', one indeed needs the expressions of $a = a(u, v, w)$, $G = G(u, v, w)$, and $\varphi = \varphi(u, v, w)$. The inversion of Eqs. (3.8), (3.9) and (3.10) can be easily made. Thus, for example, on choosing the special values

$$k_1 = 3, k_2 = 0, k_3 = -3, k_1' = 0, k_2' = 3/2, k_3' = -3/2, \tag{3.13}$$

we find

$$a = e^u, \tag{3.14}$$

$$G = \frac{w^2 e^{3u}}{v}, \tag{3.15}$$

$$\varphi = e^{-3u/2} \sqrt{v}. \tag{3.16}$$

The transformed Lagrangian is therefore

$$L' = L'(v, w; \dot{u}, \dot{v}, \dot{w}) = \frac{9}{8} v \dot{u}^2 - \frac{3}{4} \dot{u}\dot{v} + \frac{1}{8v}\dot{v}^2 - \frac{1}{4\pi w^2}\dot{u}\dot{v} + \frac{1}{24\pi v w^2}\dot{v}^2$$
$$+ \frac{v}{2\pi w^3}\dot{u}\dot{w} - \frac{1}{6\pi w^3}\dot{v}\dot{w} + \frac{v}{6\pi w^4}\dot{w}^2 - \frac{\Lambda_0 v}{8\pi w^2} - 9\lambda_1 v, \tag{3.17}$$

for which of course u is a cyclic coordinate, hence implying the existence of the constant of motion

$$\Sigma_0 \equiv \frac{\partial L'}{\partial \dot{u}} = 6\pi(3\dot{u}v - \dot{v}) + \frac{4v\dot{w} - 2w\dot{v}}{w^3}, \tag{3.18}$$

which can be used to get rid of $\dot{u}$. The usual procedure engenders now a new Lagrangian L'', involving only v, w and their first derivatives, i.e.

$$L'' = \frac{\dot{v}^2}{72\pi^2 v w^4} + \frac{\dot{v}^2}{24\pi v w^2} - \frac{\dot{v}\dot{w}}{18\pi^2 w^5} + \frac{v\dot{w}^2}{18\pi^2 w^6} - \frac{v\dot{w}^2}{6\pi w^4} + \frac{\Lambda_0 v}{8\pi w^2} + 9\lambda_1 v + \frac{\Sigma^2}{288\pi^2 v}. \tag{3.19}$$

6

Unfortunately, the resulting Euler–Lagrange equations are virtually unmanageable. We have tried also different choices of the transformation, without any apparent advantage. Therefore, in this case it is not possible for us to achieve the general exact solution of the equation, which was instead obtained in other cases. However, we think that there is some improvement of the situation under study. Indeed, the number of degrees of freedom has been reduced from 6 to 4. Moreover, by using the constraint $E_{L''} = 0$, we may further reduce them to 3. This could allow qualitative analysis of the system.

A possible interesting subcase seems to occur when $V = 0$, i.e. a massless scalar field. In this case we have that the original Lagrangian L is already cyclic in the variable φ, with the obvious symmetry $X_1 = \partial/\partial\varphi$.

One might therefore think that another symmetry X_2 can make it possible to obtain *two* cyclic variables, reducing thus the number of degrees of freedom to only two. In fact, by further constraining Λ to be a constant, we obtain the same vector field as above. Unfortunately, the two fields do not commute, so that we cannot obtain two cyclic variables with one and the same change of variables. Indeed, the procedure used to obtain the solution is not exhaustive of the possibilities, so that there is some room left for further investigations.

IV. FIXED POINT SOLUTION

One possibility to gain some information about the new situation is offered by a qualitative study of the new equations. In particular, one can try to find the fixed points of the new system and study what happens in their neighbourhood.

Starting from the new Lagrangian (3.19) it is possible to find the Euler–Lagrange equations, which are of course two and of second-order. According to the standard procedure, we want now pass to a first-order system. It is however possible to reduce by one the number of the equations, exploiting the conservation of the Energy function, which in this case must be set to zero. This yields an algebraic equation for one of the variables, and we choose the one for $\dot{v}$, which, being of second degree, gives two solutions. We have checked, however, that both lead eventually to the same result. After some algebra, and discarding the solutions which are unphysical, we obtain one interesting fixed point, for the values

$$\dot{w} = 0 \quad ; \quad v_f = \frac{\Sigma_0}{6\pi\sqrt{72\lambda_1 - 3\Lambda_0}} \quad ; \quad w_f = \sqrt{\frac{-\Lambda_0}{144\pi\lambda_1 - 3\pi\Lambda_0}}. \tag{4.1}$$

Now $\lambda_1 > 0$ since we want $G > 0$, thus the only possibility to get a real w_f is $\Lambda_0 < 0$. Unfortunately, the point is degenerate and the system cannot be linearized around it. The only thing we can do easily is to write down the solution at the point. Written in physical variables, this reads as

$$a = e^{u_f t}, \tag{4.2}$$

$$G = \frac{w_f^2 e^{3u_f t}}{v_f}, \tag{4.3}$$

$$\varphi = e^{-3u_f t/2}\sqrt{v_f}, \tag{4.4}$$

where $u_f = \Sigma_0/18\pi v_f$. We see that a and G grow exponentially, while φ decreases accordingly. As we said, we cannot say if this solution is an attractor, but it seems interesting that we have obtained a solution of inflationary type.

V. CONCLUDING REMARKS

The scalar-field-dominated cosmological model described by the Lagrangian function in Eq. (2.2) can be indeed seen as equivalent to a *standard* gravity model with *two* non-interacting scalar fields, *but* with a conformal factor multiplying a part of the Lagrangian. It is therefore necessary to achieve a clarification of open questions like, for instance, the fact that two different forms of the function $\Lambda = \Lambda(G)$ are found in two different investigations: for the pure-gravity regime ($\Lambda(G) \sim G^{2(J-2)}$, with J a parameter linked to μ) and the scalar-field-dominated one ($\Lambda(G) \sim -G$), respectively examined in Ref. [17] and here. Such two functions become comparable only upon choosing $J = 5/2$ (equivalent to take $\mu = 8/3$) in the pure gravity regime, which is indeed relevant. That peculiar value for the interaction parameter is in fact found as the common one in the pure-gravity [17] and matter-dominated cases [20] when $\Lambda G = constant$, while here we have got the completely different value $\mu = 2/3$, fixed whatever is the functional expression of ΛG. Thus, we find contradictions deserving further work for clarification.

If $V \sim \varphi^4$, in Ref. [3] power-law solutions for the scale factor $a = a(t)$ were shown to satisfy the cosmological equations. Here, the use of the Noether Symmetry Approach has in principle generalized that model and its assessment. We have in fact discovered that, even though the Noether Symmetry Approach is now partially ineffective, we are anyway left (see Eq. (3.5)) with a linear relationship between Λ and G. This is new and indeed a little

surprising in a context referring to the non-perturbative renormalizability of the theory. We should also note, in particular, the relevance of a quadratic form of the scalar field potential, and that the coupling parameter μ and the spatial curvature $\mathcal{K}$ are fixed once and for all by the method. This case is, therefore, well worth of deeper investigations in future work and, even if exact integration of the cosmological equations has to be postponed, we are surely left with new and unexpected suggestions.

In conclusion, we have to stress that we do not propose any physical interpretation of the kind of cosmic era here examined and its more appropriate location in the general evolutionary picture of the universe. This is mainly due to still lacking information on the complete paradigm into which this kind of analysis could be properly inserted.

Furthermore, as a final remark, let us just point out that the procedure adopted in this paper does not work at all (but for other reasons) with a Lagrangian where the matter term is that for radiation, $L_m \equiv Da^{-1}$ (with $\gamma = 4/3$), and other methods have to be worked out so as to solve the cosmological equations in the radiation-dominated period.

Acknowledgments

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